Math 60 9.9 The Complex Number System (2nd day of two)

Objectives 5) Finding the complex conjugate

6) Dividing complex numbers

$\frac{\text{2 terms}}{\text{2 terms}} \Rightarrow$ multiply by $1 = \frac{\text{conjugate}}{\text{conjugate}}$

$\Rightarrow$ denominator becomes a single real number

7) Dividing complex numbers

$\frac{\text{2 terms}}{\text{1 term}} \Rightarrow$ divide each term separately.

8) Powers of i

Multiply

① $(4+3i)(4-3i)$ ← Notice: These two complex numbers both have imaginary parts.

$= 16 - 12i + 12i - 9i^2$ FOIL

$= 16 - 9(-1)$

$= 16 + 10$

$= \boxed{26}$ ← But when we complete the multiplication, the result is a real number because the middle terms cancel out.

$$\left\{ \begin{array}{l} \text{It's essentially a difference of squares} \\ (x-y)(x+y) = x^2 + xy - xy - y^2 \\ \qquad\qquad\quad = x^2 - y^2 \end{array} \right\}$$

Because this pattern results in a real result, these two complex numbers, $4+3i$ and $4-3i$, are related to each other in a special way.

They are called <u>complex conjugates.</u>

$4+3i$ is the complex conjugate of $4-3i$

$4-3i$ is the complex conjugate of $4+3i$.

<u>Notice</u>: The real parts are exactly the same. The imaginary parts have opposite signs.

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Find the complex conjugate of each number.

(2) $-3+5i$

complex conjugate is $\boxed{-3-5i}$

same real part ⇑ opposite imaginary part ⇑

(3) $4i$

$4i = 0+4i$

complex conjugate is $0-4i = \boxed{-4i}$

(4) 2

$2 = 2+0i$

complex conjugate is $2-0i = \boxed{2}$ ← complex conjugate is pretty boring for purely real numbers.

An expression which has i in the denominator is not simplified --

Ex: $\dfrac{-3+i}{5+3i}$ actually means $\dfrac{-3+\sqrt{-1}}{5+\sqrt{-9}}$.

Though this denominator is complex, it can been seen as a square root.

Just as we rationalized denominators in 9.6, we want to remove this i from denominator also.

We will do this using the same techniques as 9.6, only with i.

* CAUTION * Though this process looks like rationalizing AND we do almost no dividing in the process, the instruction will say: "Divide".

Math 60 9.9 – 2nd

Divide.

⑤ $\dfrac{-3+i}{5+3i}$

Goal: Rewrite as an equivalent expression with a purely real number in denominator

Method: Multiply by $1 = \dfrac{\text{complex conjugate of denom}}{\text{complex conjugate of denom}}$

$$= \frac{-3+i}{5+3i} \cdot \frac{5-3i}{5-3i}$$

step 1: Find complex conjugate of denom.

$$= \frac{(-3+i)(5-3i)}{(5+3i)(5-3i)}$$

Multiply by 1.
dd parentheses

$$= \frac{-15+9i+5i-3i^2}{25-15i+15i-9i^2}$$

step 2: FoIL numerator
FoIL denominator

$$= \frac{-15+14i-3(-1)}{25-9(-1)}$$

step 3: Combine
Simplify $i^2=-1$.

$$= \frac{-15+3+14i}{25+9}$$

$$= \frac{-12+14i}{34}$$

* check *
If you don't have a single real number in the denominator, try again.

$$= \frac{-12}{34} + \frac{14}{34}i$$

$$= \boxed{\frac{-6}{17} + \frac{7}{17}i}$$

step 4: Divide each term by the denominator and reduce the resulting fractions.

This is a+bi form, a clearly separate real part $\left(-\frac{6}{17}\right)$ and imaginary part $\left(\frac{7}{17}i\right)$.

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Divide.

6) $\frac{3+4i}{2i}$

We have options because the denom has only one term.

Option 1: Do it the same way, mult by 1

$\frac{3+4i}{2i} \cdot \frac{-2i}{-2i}$

complex conjugate has same real part but opposite imaginary part

$= \frac{-2i(3+4i)}{-4i^2}$ distribute

$= \frac{-6i-8i^2}{-4(-1)}$ simplify $i^2=-1$

$= \frac{-6i-8(-1)}{4}$ divide each term

$= -\frac{6}{4}i + \frac{8}{4}$ reduce

$= -\frac{3}{2}i + 2$ write a+bi form

$= \boxed{2 - \frac{3}{2}i}$

Option 2: Multiply by $1 = \frac{i}{i}$ only

$\frac{(3+4i)}{2i} \cdot \frac{i}{i}$

$= \frac{3i+4i^2}{2i^2}$

$= \frac{3i+4(-1)}{2(-1)}$

Cont ⇒

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$$= \frac{3i - 4}{-2}$$

$$= \frac{3}{-2}i - \frac{4}{-2}$$

$$= -\frac{3}{2}i + 2$$

$$= \boxed{2 - \frac{3}{2}i}$$

Option 3: Divide each term by $2i$.

$$\frac{3+4i}{2i}$$

$$= \frac{3}{2i} + \frac{4i}{2i}$$

$$= \frac{3}{2i} \cdot \frac{i}{i} + 2$$

still have to multiply the 1st term by $\frac{i}{i}$

$$= \frac{3i}{2i^2} + 2$$

$$= \frac{3i}{2(-1)} + 2$$

$$= \frac{3i}{-2} + 2$$

$$= \boxed{2 - \frac{3}{2}i}$$

(7) Evaluate powers of i

a) $i^2 = \boxed{-1}$

b) $i^4 = (i^2)(i^2) = (-1)(-1) = \boxed{1}$

c) $i^3 = (i^2)(i) = -1(i) = \boxed{-i}$

d) $i^5 = (i^4)(i) = 1 \cdot i = \boxed{i}$

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e) $i^6 = i^4 \cdot i^2 = 1 \cdot (-1) = \boxed{-1}$

f) $i^7 = i^4 \cdot i^3 = 1(-i) = \boxed{-i}$

g) $i^8 = i^4 \cdot i^4 = 1 \cdot 1 = \boxed{1}$

* MEMORIZE
  $i^1 = i$
  $i^2 = -1$
  $i^3 = -i$
  $i^4 = 1$

see a pattern?

$$\begin{bmatrix} i \\ i^5 \\ i^9 \\ i^{13} \end{bmatrix} \quad \begin{bmatrix} i^2 \\ i^6 \\ i^{10} \\ i^{14} \end{bmatrix} \quad \begin{bmatrix} i^3 \\ i^7 \\ i^{11} \\ i^{15} \end{bmatrix} \quad \begin{bmatrix} i^4 \\ i^8 \\ i^{12} \\ i^{16} \end{bmatrix}$$

$\Uparrow$	$\Uparrow$	$\Uparrow$	$\Uparrow$
all i	all -1	all $-i$	all $+1$

Evaluate

(8) i^{27}

$= i^{24} \cdot i^3$

$= (1)(-i)$

$= \boxed{-i}$

step 1: find nearest power of 4 that is less than exponent and rewrite using exponent laws

$a^{n+m} = a^n \cdot a^m$

step 2: Recall that $i^n = 1$ when n is a multiple of 4.

step 3: simplify other part from memory.

(9) i^{38}

$= i^{36} \cdot i^2$

$= (1)(-1)$

$= \boxed{-1}$

(10) $i^{40} = \boxed{1}$